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(NASA-TM-89779) ON THE TRANSFORMATION OF SPACE-TIME COORDINATES FROM STATIONARY TO MOVING SYSTEMS (Friedman (Morris D.)) 31 p

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1. On the Transformation of Space-Time Coordinates from Stationary to Moving Systems

P. FRANK, H. ROTHE

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The equations of transformation which relate the space-time coordinates (x, y, z, t) of a stationary system to that of a moving system (x', y', z', t') whose speed q is constant in direction and magnitude, have obtained such importance in present-day physics that it may well be worth our while to undertake an accurate examination of what fundamental assumptions of a physical (or other) nature are necessary in order to derive the form of these equations. According to the theory of relativity, they are given by the Lorentz transformation. If we designate the velocity of light in a vacuum by c and choose the coordinate system such that the stationary and the moving systems coincide at time 0 , and the moving system then continues to propagate in the x direction, the Lorentz transformation is of the well known form:

$$(1)\qquad \begin{cases} t' = \dfrac{1}{\sqrt{1-\dfrac{q^2}{c^2}}}\left(t - \dfrac{q}{c^2}x\right), \\[2ex] x' = \dfrac{1}{\sqrt{1-\dfrac{q^2}{c^2}}}\left(-qt + x\right). \end{cases}$$

In the limit case $c = \infty$, these equations contain the Galileo transformation:

$$(2)\qquad t' = t, \qquad x' = -qt + x .$$

The derivation of equations (1) in its present form is due to A. Einstein, [1] and essentially rests on the following assumptions:

α)* When c is the value of the velocity of light with respect to a system at rest, the value of the velocity of light with respect to every system moving uniformly relative to the first in a rectilinear fashion must also equal c for all directions of propagation. Mathematically, this corresponds to the postulate that the relations

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \quad \text{and} \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

be derivable from one another as a consequence of the transformation equations.

β) The transformation equations must be linearly homogeneous with respect to coordinates; then their coefficients can depend only on q.

γ) When $-q$ is substituted for q, the transformation must change to its inverse (i.e., solved for x, y, z, t).

δ) The contraction experienced by the lengths due to their motion must depend not on the sign of q but only on its magnitude.

We should now like to demonstrate that the number of these assumptions may be greatly limited, and especially that α), the postulate which appears to be most important, namely, the postulate of the constancy of the velocity of light in stationary or mivng systems, may be discarded.

Instead, our derivation is based on only the following two suppositions:

A. When we consider q a variable parameter, the transformation equations form a linear homogeneous group.

B. The contraction in the lengths is not to depend on the sign of q, but only on its magnitude.

The group characteristic of the transformation equation required in A must necessarily be postulated if there is above all to exist a type of transformation equation valid for all velocities q. For if the equations did not form a group, then the combination of two transformations, i. e., the transition from one system to one moving with the help of an intermediate system, would lead to equations of quite a different form than the original ones.

We now proceed in such a manner that we first establish the most general transformation equations that satisfy postulate A. We then obtain all those which also fulfill postulate B by specializing the former. As a result, only equations remain which either do not lead to any contraction at all or which coincide with those of Lorentz (1). The former equations form a new type of transformation equations ("Doppler transformation") which contain the Galileo transformation (2) as a special case.

We have already published a part of the statements and formulas used here for our proof in another paper [2], "On a Generalization of the Principle of Relativity and the Mechanics Applicable Thereto".

Among these is the theorem of the addition of velocities for the most general transformation equation satisfying postulate A .*

Mr. V. Ignatowsky [3] has already attempted to restrict Einstein's premises to a smaller number.

When one also expresses his implied suppositions, one can render the contents of his paper as follows: he avoids the assumption α) (constancy of the velocity of light), but retains Einstein's postulate, γ) in addition to our assumptions. Furthermore, he immediately makes use of all the premises and does not establish the most general transformation equation satisfying postulate A , from which alone the position of the Lorentz transformation within all the others would become clearly manifest.

This paper is organized, as follows: We make the following a priori assumption:

$$y' = y, \quad z' = z,$$

since the proof of these equations, while relatively simple, would merely encumber our train of thought with avoidable clumsiness. We examine, then, only the transformation of x and t .

In Section I, we briefly restate the concepts and premises used here from the Theory of Transformation Groups.**

In Sections, II , and III , we shall make use of these premises in the context of the transformation equations determined by supposition A. In, Section IV , we shall introduce a parameter *** q which has the properties of a velocity. This leads to the theorem of the addition of velocities. Examples relating to the foregoing developments are given in, Section V . Finally, in , Section VI, we determine the form of the most general of the transformation equations satisfying postulate A , and, especially, the contraction as a function of the velocity q introduced in, Section IV. In, Section VII, we then apply our postulate B to these, and so obtain all the equations which satisfy our system of premises.

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* Equation (12) and the one following it (not numbered) in [2], p. 619.

** We shall refer at all times to the elementary representation of the Theory of Groups in the book, "Vorlesungen über kontinuierliche Gruppen" ("Lectures on Continuous Groups") by S. Lie and G. Scheffers, Leipzig, 1893. [4] .

*** Compare also, Ph. Frank and H. Rothe, [2], p. 618.

I .

1. Let $t, x, p,$ be three variables such that t, x , are right-angle coordinates of a point P in a t, x-plane, and p is a parameter. Further, let

$$\varphi(t, x, p), \quad \psi(t, x, p)$$

be two single-valued, continuous and differentiable functions of the three arguments * t, x, p for which the functional determinant

$$\frac{\partial(\varphi,\psi)}{\partial(t,x)} \equiv \begin{vmatrix} \frac{\partial\varphi}{\partial t} & \frac{\partial\psi}{\partial t} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\psi}{\partial x} \end{vmatrix} \tag{4}$$

does not vanish identically; moreover,

$$\frac{\partial\varphi}{\partial p} \equiv 0, \quad \frac{\partial\psi}{\partial p} \equiv 0 \tag{5}$$

may not hold simultaneously.

When a fixed value is imparted to the parameter p , a second pair of values t' , x' is associated with each pair of values t, x by means of the two equations

$$t' = \varphi(t,x,p), \quad x' = \psi(t,x,p) \tag{6}$$

This association is called a "transformation" and may be designated by T_p . In its geometric interpretation, transformation T_p signifies a "point-by-point mapping" of the t , x-plane onto a t', x' -plane which (as will be presupposed in the following) may coincide with the t, x-plane. Hence, to begin with, we relate the coordinates t', x' of the transformed points P' to the same coordinate system as the coordinates t, x of the original points P .

2. If the parameter p runs continuously through the entire number-sequence or a definite interval of it, we obtain an aggregate $\mathfrak{G}$ of ∞^1 transformations T_p , each, of which, corresponds to a definite value of p ; this aggregate is also designated as a one-parameter (continuous) aggregate of transformations.

If T_p' is a second transformation of the aggregate $\mathfrak{G}$ which belong to the parameter P' and which transforms the pair t', x' into t'', x'' so that then

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* If necessary, the three variables t, x, p must be confined to a definite domain of the t, x, p-manifold; to which every system of values t, x, p worthy of further consideration must belong.

(7) $$t'' = \varphi(t',x',p'), \quad x'' = \psi(t',x',p')$$

holds, by eliminating t', x' from (6) and (7), we then obtain equations

(8) $$\begin{cases} t'' = \varphi\,(\varphi(t,x,p),\ \psi(t,x,p),\ p'), \\ x'' = \psi\,(\varphi(t,x,p),\ \psi(t,x,p),\ p'), \end{cases}$$

which represent a transformation T which transforms t, x directly into t'', x'' and is called the "product" of the two transformations T_p and $T_{p'}$; one writes

(9) $$T = T_p T_{p'}$$

where the sequence in which the two transformations T_p and $T_{p'}$ are to be carried out is given by the order of the factors of the product. In general,

(10) $$T_{p'} T_p \neq T_p T_{p'},$$

i.e., the commutative law is not valid for the composition of transformations.

3. In general, the product T of the two transformations T_p and $T_{p'}$ of $\mathfrak{G}$ will be a transformation which does not belong to the aggregate $\mathfrak{G}$. However, if the product of any two transformation from $\mathfrak{G}$ always appears as a transformation from $\mathfrak{G}$, it is said that the transformations of the aggregate $\mathfrak{G}$ possess the group characteristic. In that case, i.e., the associative law is valid for products of three (and also for any number of) factors.

If $\mathfrak{G}$ possesses the group characteristic, i.e., if T belongs to $\mathfrak{G}$, then equations (8) must take the form:

(12) $$t'' = \varphi(t,x,p''), \quad x'' = \psi(t,x,p'')$$

where

(13) $$p'' = \pi(p,p')$$

is a function of p and p' only.

We can now say that the transformations of a set $\mathfrak{G}$ form a group $\mathfrak{g}$ when the following conditions are fulfilled:

A. The transformations of $\mathfrak{G}$ possess the group characteristic.

B. There exists a value of the parameter $p = p_0$ for which

(14) $$\varphi(t,x,p_0) \equiv t, \quad \psi(t,x,p_0) \equiv x$$

The transformation T_{p_0} which belongs to this parameter and is represented by the equations

(15) $$t' = t\,, \quad x' = x$$

thus leaves every -pair t,x unchanged and is called the identity transformation.

C. For each transformation T_p there exists a second in $\mathfrak{G}$, which when combined with T_p in some sequence yields the identity transformation T_{p_o}. This second transformation is called the inverse transformation to T_p and is designated by T_p^{-1}, so that

$$T_p T_p^{-1} = T_p^{-1} T_p = T_{p_o} . \tag{16}$$

The inverse transformation of T_p is found by solving equation (6) for t and x, which can always be done since the functional determinant (4) does not vanish identically. As a transformation of set $\mathfrak{G}$, a parameter $\bar{p}$, which is solely a function of p, corresponds to the inverse transformation T_p^{-1} of T_p. This value is found with the help of condition

$$T_p^{-1} = T_{\bar{p}} \tag{17}$$

According to (13), the two values p and $\bar{p}$ satisfy the equation

$$\pi(p, \bar{p}) = p_o . \tag{18}$$

The group $\mathfrak{G}$ is called "one-parameter" because it consists of ∞^1 transformations T_p.

4. If p is considered as a variable to be transformed in (13), and p' as a parameter (or vice versa), then this equation defines a one-parameter set of transformations which also form a group $\mathfrak{P}$ if p" is the transformed variable. The group $\mathfrak{P}$ is designated as the parameter group of $\mathfrak{G}$.

5. If δp is an infinitesimal quantity, then transformation which belong to the parameter

$$p = p_o + \delta p \tag{19}$$

differs infinitesimally from the identity transformation. This is the infinitesimal transformation of the group, and it transforms a point $P = (t,x)$ into an infinitely close neighboring point P' which has the coordinates

$$t' = t + \delta t, \qquad x' = x + \delta x \tag{20}$$

where

$$\delta t = \tau(t,x)\delta p , \qquad \delta x = \xi(t,x)\delta p \tag{21}$$

when we let

(22) $$\varphi'_p(t,x,p_0) \equiv \tau(t,x), \quad \psi'_p(t,x,p_0) \equiv \xi(t,x).$$

In equations (21), which define the infinitesimal transformation, δp should be replaced by $\kappa \cdot \delta p$ (when κ is a constant different from 0) without essentially changing any property of group $\mathcal{G}$. If two infinitesimal transformations which are mutually dependent in this manner are regarded as being identical, then every one-parameter group contains only a single infinitesimal transformation. Conversely, every arbitrary infinitesimal transformation (21) generates a particular one-parameter group. The final equations (6) are found by integration of the simultaneous system

(23) $$\frac{dt'}{\tau(t',x')} = \frac{dx'}{\xi(t',x')} = dp$$

with the initial conditions:

(24) $$t' = t, \quad x' = x \quad \text{for } p = p_0$$

6. If now x is considered as a function of t :

(25) $$x = f(t),$$

a curve Γ in the t , x-plane is obtained; this is transformed by means of transformation (6) into another curve Γ with the equation

(26) $$x' = f_1(t')$$

If one sets

(27) $$w = \frac{dx}{dt} = f'(t), \quad w' = \frac{dx'}{dt'} = f_1'(t'),$$

then

(28) $$w' = \frac{\psi'_t(t,x,p) + \psi'_x(t,x,p) \cdot w}{\varphi'_t(t,x,p) + \varphi'_x(t,x,p) \cdot w},$$

for which we may write briefly

(29) $$w' = \chi(t,x,w,p)$$

This is the transformation of w belonging to (6) .

For $p = p_0$, according to (14), we obtain

(30) $$\begin{cases} \varphi'_t(t,x,p_0) \equiv 1, & \varphi'_x(t,x,p_0) \equiv 0, \\ \psi'_t(t,x,p_0) \equiv 0, & \psi'_x(t,x,p_0) \equiv 1, \end{cases}$$

hence:

(31) $$\chi(t,x,w,p_o) \equiv w \ ,$$

i.e.,

(32) $$w' = w \ .$$

For $p = p_o + \delta_p$, we have

(33) $$w' = w + \delta w$$

and, we obtain for the infinitesimal transformation of w ,

(34) $$\delta w = [\xi_t' + (\xi_x' - \tau_t')\, w - \tau_x'\, w^2]\delta p$$

or, more briefly:

(35) $$\delta w = \eta(t,x,w)\ .\ \delta p$$

The equations (6) and (28) together again constitute a group $\mathcal{G}_1$ of transformations which transform the variables t, x, w into t', x', w' . This group $\mathcal{G}_1$ is called the first extended group ; its infinitesimal transformation is given by equations (21) and (34), and one may find from it the final equations (6) and (28) of the group by integration of the simultaneous system

(36) $$\frac{dt'}{\tau(t', x')} = \frac{dx'}{\xi(t',x')} = \frac{dw'}{\eta(t', x', w')} = dp$$

with the initial conditions

(37) $$t' = t, \quad x' = x, \quad w' = w \quad \text{for} \quad p = p_o \ .$$

II.

7. Let us now select a coordinate system S consisting of a fixed straight line, the x-axis, and a fixed point O , its origin. We imagine a fixed measuring rod placed on the x-axis with the zero-point O and we imagine a clock attached to each point on the rod.

If we then observe the motion of a material point M along the x-axis, there corresponds to each position, a definite pair of values t, x, namely, a definite position of the pointer of that clock which belongs to the point on the axis with which M coincides, and a definite division of the measuring rod. Every definite motion is represented by a definite equation, (25), and the velocity w is then given by the first of the equations (27).

If we interpret the quantities t, x as coordinates of a point P of the t, x-plane, then there corresponds to each position of M a definite point P , which is called the "space-time point" of this position. We call t, x the space-time coordinates measured in the system S. The entire motion of M is represented by a continuous sequence of space-time points, i.e., by means of a curve Γ, the equation of which is (25) and which is called the "world line of this motion". The velocity w at time (time-point) t is equal to the direction coefficient of the tangent of the world line curve at the space-time point P. The world line curve corresponding to a uniform motion of M is a straight line.

8. Besides the system S, let us also examine, on the same straight line, a singly infinite set of other systems S' (i.e., of other measurements of length and time), each of which is associated with a certain value of a parameter p in such a manner that the various values of p correspond to different systems S'. An arbitrary space-time point P which possesses the space-time coordinates t, x in the system S should then also possess definite space-time coordinates t', x' in each of the systems S' which would depend only on t, x, and p, that is, the space-time coordinates t, x and t', x' of P are to be related by equations of the form (6) with respect to S and S'. The quantities t' and x' are called the space-time coordinates of P measured in system S'. Associated with every space-time point, there are then an infinite number of pairs t', x' corresponding to the infinite number of values of p. These pairs are derived from t, x by means of a one-parameter set $\mathfrak{G}$ of transformations (6).*

If we carry out two transformation of the set $\mathfrak{G}$ in succession by passing from the system S to a second system S' by means of the transformation equation (6) and then use equation (7) in order to transform to a third system S", then the product of the two transformations, i. e., transformation (8) which serves as a direct intermediate step from S to S", must also belong to the set $\mathfrak{G}$; that is, the set $\mathfrak{G}$ is to have the group characteristic.

We assume further that the original system S itself occurs among the systems S'. If then the parameter p_o is associated with it, then the equation (6) must be transformed into equation (15) for $p = p_o$; i.e., the set $\mathfrak{G}$ must contain the identity transformation.

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* The final portion of Section II, beginning at this point, does not contribute to the understanding of main argument of the paper, but merely serves to render plausible our postulate A.

Finally, we suppose that there is in the set $\mathfrak{G}$ an inverse for each transformation; then, for each parameter p, there is a second, $\bar{p}$, such that p and $\bar{p}$ satisfy equation (18). In that case, the transformations of the set $\mathfrak{G}$ form a one-parameter group $\mathcal{G}$, and we may combine the above three asumptions into a single one by making the following supposition:

The transformations (6) which govern the transition of the space-time coordinates t, x measured in the original system S to the space-time coordinates, t', x' measured in a system S', form a one-parameter group with parameter p.

9. In order to determine the group $\mathcal{G}$ further, we now make the following further assumptions:

A. Every motion of a material point M which is uniform with respect to the system S at rest, must also be uniform with respect to each of the moving systems S'. Hence, if the world line curve Γ of a motion of M with respect to S is a straight line, then the world line curve Γ, of the same motion with respect to S' must also be a straight line; i.e., transformations of the group $\mathcal{G}$ must be of such a nature that they transform straight lines into straight lines.

However, the only transformations of this type are projective transformations [4], that is, those which have equations (6) of the following special form:

$$(38)\qquad \begin{cases} t' = \dfrac{a_{11}(p) + a_{12}(p)\,x + a_{13}(p)}{a_{31}(p) + a_{32}(p)x + a_{33}(p)}\,, \\[2ex] x' = \dfrac{a_{21}(p)t + a_{22}(p)x + a_{23}(p)}{a_{31}(p)t + a_{32}(p)\,x + a_{33}(p)}\,. \end{cases}$$

The group $\mathcal{G}$ is then designated as a one-parameter projective group.

B. Every space-time point which has finite coordinates t,x with respect to a system S ought also to have finite coordinates t', x' with respect to each system S'. From this it follows [4] that we must have in equations (38)

$$(39)\qquad a_{31}(p) \equiv 0, \qquad a_{32}(p) \equiv 0$$

If we designate

$$(40)\qquad \frac{a_{ik}(p)}{a_{33}(p)} \quad (i = 1,2;\ \ k = 1,2,3)$$

again, by $a_{ik}(p)$, equation (38) takes the form

(41) $$\begin{cases} t' = a_{11}(p)\, t + a_{12}(p)\, x + a_{13}(p) \,, \\ x' = a_{21}(p)\, t + a_{22}(p)\, x + a_{23}(p) \,. \end{cases}$$

Transformations (41) leave the infinitely distant straight line of the t, x-plane invariant, and are called affine. The group $\mathcal{G}$ is then called affine, or generally linear.

C. Finally, the null-point of a space-time measurement must be the same for all systems, i.e., from

$$t = 0, \quad x = 0$$

must always follow

$$t' = 0, \quad x' = 0$$

Then

(43) $$a_{13}(p) \equiv 0, \quad a_{23}(p) \equiv 0$$

must hold, so that the equations (41) are transformed into the following:

(42) $$t' = a_{11}(p)t + a_{12}(p)x, \quad x' = a_{21}(p)t + a_{22}(p)x$$

Now, then, t', x' are linear homogeneous functions of t, x with coefficients which are solely functions of the parameter p. The group $\mathcal{G}$ is now designated as a one-parameter linear homogeneous group, and its transformations leave the infinitely distant straight line of the t, x-plane, and moreover, its zero-point, invariant [4].

It is hardly worth mentioning here that the coefficients $a_{ik}(p)$ must not be selected arbitrarily but subject to certain conditions if the transformations are to form groups. In the following, we shall occupy ourselves with the determination of the form of these coefficients.

III.

10. We may now summarize all the suppositions which we have made concerning transformation (6) in the following manner:

The transformations (6) which represent the relationship between the space-time coordinates with respect to the original system S and the system S' constitute a one-parameter linear homogeneous group with the parameter p.

In order for equations (43) to be transformed for the parameter value $p = p_o$ into equations (15), which represent the identity transformation,

$$(44)\qquad \begin{cases} a_{11}(p_o) = 1 \ , \quad a_{12}(p_o) = 0, \\ a_{21}(p_o) = 0 \ , \quad a_{22}(p_o) = 1 \end{cases}$$

must hold. For the parameter value $p = p_o + \delta_p$, we there fore obtain the coefficients

$$(45)\qquad \begin{cases} a_{11}(p_o + \delta_p) = 1 + a_{11}'(p_o)\delta p \ , \\ a_{21}(p_o + \delta_p) = \qquad a_{21}'(p_o)\delta p \ , \\ \\ a_{12}(p_o + \delta p) = \qquad a_{12}'(p_o)\delta p, \\ a_{22}(p_o + \delta p) = 1 + a_{22}'(p_o)\delta p, \end{cases}$$

and if we let

$$(46)\qquad \begin{cases} a_{11}'(p_o) = \alpha_{11}, \quad a_{12}'(p_o) = \alpha_{12} \ , \\ a_{21}'(p_o) = \ _{21}, \quad a_{22}'(p_o) = \alpha_{22} \ , \end{cases}$$

the equations for the infinitesimal transformation result [compare No. 5, equations (19), (20), (21), (22)] in the form

$$(47)\qquad \delta t = (\alpha_{11} t + \alpha_{12} x)\delta p, \qquad \delta x = (\alpha_{21}\, t + \alpha_{22} x)\delta p,$$

so that for the linear homogeneous group (43), we have

$$(48)\qquad \tau(t,x) \equiv \alpha_{11} t + \alpha_{12} x, \qquad \xi(t,x) \equiv \alpha_{21} t + \alpha_{22} x$$

The coefficients α_{11}, α_{12}, α_{21}, α_{22} may be chosen arbitrarily; only their relationships are essential. Hence there are ∞^3 infinitesimal transformations (47), and each of them generates a definite one-parameter linear homogeneous group (43).

11. If we now examine a certain transformation of $\mathcal{G}$, i.e., if we impart some fixed value to the parameter p , by differentiating the equation (43), then we obtain equations

$$(49)\qquad dt' = a_{11}(p) \cdot dt + a_{12}(p) \cdot dx, \qquad dx' = a_{21}(p) \cdot dt + a_{22}(p) \cdot dx \ ,$$

from which it becomes evident that the differentials dt, dx are transformed in the same manner as the two finite quantities t, x ; that therefore the two pairs of quantities t, x and dt, dx undergo the co-gradient (kogredient) transformations (43) and (49).

From equations (49) follows

(50) $$\frac{dx'}{dt'} = \frac{a_{21}(p) \cdot dt + a_{22}(p) \cdot dx}{a_{11}(p) \cdot dt + a_{12}(p) \cdot dx}$$

and, therefore, because of (27) :

(51) $$w' = \frac{a_{21}(p) + a_{22}(p) \cdot w}{a_{11}(p) + a_{12}(p) \cdot w} \ .$$

This equation, which gives the transformation of the velocity w into w', now takes the place of equation (28) and, along with equations (43), represents the first extended group $\mathcal{G}_1$. Of special importance is the circumstance that in the case of the linear group, w' is only a function of w and p, but does not depend on t and x.

The infinitesimal transformation of the velocity w is finally obtained by means of (34) and (38) in the form:

(52) $$\delta w = - \ [- \alpha_{21} + (\alpha_{11} - \alpha_{22})w + \alpha_{12}w^2] \ \delta p \ ,$$

from which it may be concluded that the function $\eta(t,x,w)$ in (35) is now free of the quantities t and x. By means of this infinitesimal transformation, the velocity w is transformed, using (33), into $w' = w + \delta w$ and thus remains unchanged if and only if

(53) $$\delta w = 0.$$

This holds true for those velocities which are roots of the quadratic equation

(54) $$- \alpha_{21} + (\alpha_{11} - \alpha_{22})w + \alpha_{12}w^2 = 0.$$

These remain unchanged when undergoing the infinitesimal transformation, and, therefore (and as we shall show at the end of No. 12) when undergoing any finite transformation of the group $\mathcal{G}$. In the following, we shall call them preferred (ausgezeichnet) velocities, and we make this supposition:

The velocity $w = 0$ (i.e., at rest) ought not to be a preferred velocity, from which it follows that we must have

(55) $$\alpha_{21} \neq 0.$$

Due to supposition (55), the case

$$\alpha_{11} = \alpha_{22}, \qquad \alpha_{12} = \alpha_{21} = 0$$

in which equation (54) is satisfied identically and according to which each velocity w would be a preferred one, is excluded at this point. But in all the other cases, we have only two preferred velocities, which we shall designate by c_1 and c_2, namely,

(56) $$c_1 = \frac{\alpha_{22} - \alpha_{11} + \sqrt{\theta}}{2\alpha_{12}}, \qquad c_2 = \frac{\alpha_{22} - \alpha_{11} - \sqrt{\theta}}{2\alpha_{12}},$$

where

$$\theta = (\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}\alpha_{21}$$

The symmetric fundamental functions of the roots c_1 and c_2 are

(58) $$c_1 + c_2 = \frac{\alpha_{22} - \alpha_{11}}{\alpha_{12}}, \qquad c_1 c_2 = -\frac{\alpha_{21}}{\alpha_{12}},$$

By means of these relations, the infinitesimal transformation (52) may easily be put into the form :

(59) $$\delta w = \alpha_{21}\left(1 - \frac{w}{c_1}\right)\left(1 - \frac{w}{c_2}\right)\delta p$$

where the significance of c_1 and c_2 as preferred velocities is immediately apparent.

IV.

12. Now, in order to find the finite equations (43) and (51) of the extended group $\mathcal{G}_1$, which is generated by its infinitesimal transformation (47) and (52), we would, in accordance with No. 6 (36), have to integrate the simultaneous system

(60) $$\frac{dt'}{\alpha_{11}t' + \alpha_{12}x'} = \frac{dx'}{\alpha_{21}t' + \alpha_{22}x'} = \frac{dw'}{-[-\alpha_{21} + (\alpha_{11} - \alpha_{22})w' + \alpha_{12}w'^2]} = dp$$

with initial conditions (37). Meanwhile, we only want to determine equation (51) in this manner for the transformation of the velocity w, by making use of the circumstance that w' depends only on w and p, but not on t and x, so that we directly integrate the equation contained in the system (60),

(61) $$\frac{dw'}{-[-\alpha_{21} + (\alpha_{11} - \alpha_{22})w' + \alpha_{12}w'^2]} = dp$$

with the initial condition

(62) $$w' = w \quad \text{for} \quad p = p_0$$

separate from it. We next obtain

(63) $$\int_w^{w'} \frac{dw'}{-[-\alpha_{21} + (\alpha_{11} - \alpha_{22})w' + \alpha_{12}w'^2]} = \int_{p_0}^{p} dp,$$

and from this, evaluating the integrals and noting that the two preferred velocities c_1 and c_2 are the two zeros of the denominator in the first integral of (63):

(64) $$\frac{1}{\sqrt{\theta}} \ln (c_1, c_2, w, w') = p - p_o ,$$

where, by (c_1,c_2,w,w') is understood the cross ratio of the four values c_1,c_2, w, and w'; i.e., the expression

(65) $$(c_1,c_2,w,w') + \frac{(c_1 - w)(c_2 - w')}{(c_2 - w)(c_1 - w')} .$$

From (64), there finally follows

(66) $$(c_1, c_2, w, w') = e^{\sqrt{\theta} \cdot (p-p_o)}$$

and from this is found, by solving for w' ,

(67) $$w' = \frac{c_1 c_2\left[1 - e^{\sqrt{\theta}(p-p_o)}\right] - \left[c_2 - c_1{}^{\sqrt{\theta}(p-p_o)}\right] \cdot w}{\left[c_1 - c_2 e^{\sqrt{\theta}(p-p_o)}\right] - \left[1 - e^{\sqrt{\theta}(p-p_o)}\right] \cdot w} ,$$

with which the finite solution for the transformation of the velocity is found [compare (28) and (61)] . In (67), c_1 and c_2 may also be replaced by their values (56).*

Finally, it can be seen from equation (57) that the two preferred velocities c_1 and c_2 indeed do remain unchanged for every finite transformation of the group; therefore, that they have the same values with respect to every system S' , for if one lets

$$w = \begin{cases} c_1 \\ c_2 \end{cases} ,$$

it follows that

$$w' = \begin{cases} c_1 \\ c_2 \end{cases} = w$$

for every value of the parameter p (compare No. 11) .

13. We now wish to consider the systems S' which were introduced in Section II, No.8, as such systems which move with different constant velocities q, with respect to the original system S , designated as a system at rest. Then there is associated

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* Note that equations (64), (66) and (67) are independent of the sign associated with the quantity $\sqrt{\theta}$, for if $\sqrt{\theta}$ is replaced by $-\sqrt{\theta}$, then, by (56), the two preferred velocities c_1 and c_2 are simultaneously interchanged and the equations remain unchanged.

with each system S' a certain parameter p, as well as a definite velocity q, from which follows that there must exist a relation between p and q which we may represent in the form

(68) $$p = F(q)$$

so that the parameter p appears as a function of the velocity q.

If, using equation (68), we introduce the velocity q in equations (43) and (51) in place of the parameter p, nothing is essentially changed in group $\mathcal{G}_1$; q may then be looked upon as a new parameter of the group.

In order to define the velocity q, and hence to determine the form of the function $F(q)$, let us establish the following postulate:

When a material point M moves with a velocity $w = q$ with respect to a system at rest S, then it should have the velocity $w' = 0$ with respect to a system S' moving uniformly with a velocity q with respect to S.

This postulate states that the pair

(69) $$w = q, \quad w' = 0$$

should satisfy equation (64), so that we then have

(70) $$p - p_o = \frac{1}{\sqrt{\theta}} \ln (c_1, c_2, q, 0)$$

From this we find the desired function $F(q)$:

(71) $$p = F(q) = p_o + \frac{1}{\sqrt{\theta}} \ln \frac{c_2(c_1 - q)}{c_1(c_2 - q)}$$

and obtain, by substituting this expression in (67), the transformation equation for the velocity w, in the form

(72) $$w' = \frac{c_1 c_2 (w - q)}{c_1 c_2 - (c_1 + c_2) q + qw},$$

And, finally, using (58);

(73) $$w' = \frac{-\alpha_{21}(w - q)}{-\alpha_{21} + (\alpha_{11} - \alpha_{22}) q + \alpha_{12} qw}.$$

From (71) it follows further that the parameter p_o of the identity transformation, corresponds to the zero value of the velocity q; that therefore the system at rest S is to be regarded as that one among the systems S' which moves with velocity $q = 0$.

14. If we henceforth regard the velocity q as the parameter of our group and set

(74) $$a_{ik}(F(q)) \equiv b_{ik}(q), \quad (i,k = 1,2) \;,$$

then, in lieu of (43) and (51), we obtain, the equations:

(43a) $$t' = b_{11}(q) \cdot t + b_{12}(q) \cdot x, \quad x' = b_{21}(q) \cdot t + b_{22}(q) \cdot x$$

and

(51a) $$w' = \frac{b_{21}(q) + b_{22}(q) \cdot w}{b_{11}(q) + b_{12}(q) \cdot w} \;,$$

which now define the group $\mathcal{G}_1$. If we set $w = q$ in equation (51a), then w' must become zero regardless of the value of q . From this follows the identity:

(75) $$b_{21}(q) + q \cdot b_{22}(q) \equiv 0$$

If we now use the new equations (43a) and (51a) as the fundamental equations of the group $\mathcal{G}_1$, then the value $q = 0$ yields the identity, and therefore the value $q = \delta q$ yields its infinitesimal transformation. We may also consider it even now as given by equation (47); for by introducing the new parameter q, the values of the coefficients α_{ik} may themselves be altered, but not so their relationships - and after all, only these are essential.

By normalizing the parameter of the group, the coefficients α_{ik} are themselves given definite values, whereas up to now, only their relationships had been fixed.

Indeed, according to (45) and (46):

$$b_{21}(\delta q) = \alpha_{21}\,\delta q, \qquad b_{22}(\delta q) = 1 + \alpha_{22}\delta q \;,$$

Therefore, setting $q = \delta q$ in identity (75) and omitting members of second order in δq , we obtain :

$$(\alpha_{21} + 1)\delta q = 0 \;,$$

i. e.,

(76) $$\alpha_{21} = -1 \;,$$

since $\delta q \neq 0$. Thus the coefficient α_{21} , which had hitherto been connected only with the inequality (55), is now determined exactly.

Using (44) and (46), we further obtain the following equations for the new coefficients $b_{ik}(q)$ in (43a) and (51a):

(44a) $$\begin{cases} b_{11}(0) = 1, & b_{12}(0) = 0, \\ b_{21}(0) = 0, & b_{22}(0) = 1 \end{cases}$$

and

(46a)
$$\begin{cases} b_{11}'(0) = \alpha_{11}, & b_{12}'(0) = \alpha_{12}, \\ b_{21}'(0) = -1, & b_{22}'(0) = \alpha_{22} . \end{cases}$$

If we further substitute the value (76) in the equations (47) and (52), we obtain

(47a)
$$\delta t = (\alpha_{11} t + \alpha_{12} x)\delta q, \qquad \delta x = (-t + \alpha_{22} x)\delta q$$

for the infinitesimal transformation of group $\mathcal{G}$, and

(52a)
$$\delta w = -[1 + (\alpha_{11} - \alpha_{22})w + \alpha_{12} w^2]\delta q,$$

for the infinitesimal transformation of the velocity w, or, from (59),

(59a)
$$\delta w = -\left(1 - \frac{w}{c_1}\right)\left(1 - \frac{w}{c_2}\right)\delta q .$$

The finite equation (73) for the transformation of the velocity w transforms into

(73a)
$$w' = \frac{w - q}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12} q w}$$

and from (57) finally there results

(57a)
$$\theta = (\alpha_{11} - \alpha_{22})^2 - 4\,\alpha_{12} .$$

15. If we combine the transformation (73a), which belongs to the parameter q and which transforms w into w', with a second transformation of the same kind,

(77)
$$w'' = \frac{w' - q'}{1 + (\alpha_{11} - \alpha_{22})q' + \alpha_{12} q' w'},$$

which belongs to a parameter of value q', and transforms w' into w'', then it is a consequence of the group characteristic of our transformations that the resulting transformation, which transforms w directly into w'', must be of the form

(78)
$$w'' = \frac{w - q''}{1 + (\alpha_{11} - \alpha_{22})\, q'' + \alpha_{12}\, q'' w}$$

where the parameter value q'', by virtue of (13), is a function of q and q'.

In order to determine this function in our case, we merely have to carry out the actual combination of the two transformations (73a) and (77). We then obtain:

(79)
$$\begin{cases} w'' = \dfrac{\dfrac{w - q}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}\, q\, w} - q'}{1 + (\alpha_{11} - \alpha_{22})q' + \dfrac{\alpha_{12} q'(w - q)}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}\, q\, w}} \\[2ex] = \dfrac{w - \dfrac{q + q' + (\alpha_{11} - \alpha_{22})q\,q'}{1 - \alpha_{12}\, q\, q'}}{1 + (\alpha_{11} - \alpha_{22})\, \dfrac{q + q' + (\alpha_{11} - \alpha_{22})q\,q'}{1 - \alpha_{12}\, qq'} + \alpha_{12}\, \dfrac{q + q' + (\alpha_{11} - \alpha_{22})q\,q'}{1 - \alpha_{12}\, qq'}\, w}, \end{cases}$$

and from this, there follows, by a comparison with (78),

(80) $$q'' = \frac{q + q' + (\alpha_{11} - \alpha_{22}) q\, q'}{1 - \alpha_{12}\, q\, q'} .$$

This equation, which now determines the parameter group $\mathcal{P}$ of our group $\mathcal{G}$ (and $\mathcal{G}_1$)* expresses the addition theorem of the velocities q, where q'' signifies the velocity of a system S'' referred to the system at rest S, and where S'' moves with a velocity q' with respect to a system S' which in turn possesses the velocity q with respect to the stationary system S.

Finally, if equation (77) is to represent the inverse transformation of (73a), then,

$$w'' = w$$

hence, the resulting transformation (78) must be the identity transformation, and therefore q'' must be zero. But then, as a consequence of (80), when we designate the parameter of the inverse transformation of (73a) by $\bar{q}$, we have

(81) $$q + \bar{q} + (\alpha_{11} - \alpha_{22}) q\, \bar{q} = 0$$

and, hence,

(82) $$\bar{q} = \frac{-q}{1 + (\alpha_{11} - \alpha_{22}) q} .$$

If this value is substituted in place of q' in (77), then one obtains for the inverse transformation of (73a)

(83) $$w = \frac{q + w' + (\alpha_{11} - \alpha_{22}) q\, w'}{1 - \alpha_{12} q\, w'} ,$$

which may also be obtained directly by solving (73a) for w.

Formula (83) demonstrates that w is found from q and w' in exactly the same manner as q'' is found from q and q'. This analogy is explained without difficulty on the basis of the kinematic meaning of the two equations (80) and (83)

V.

16. Before we establish the finite equations (43a) of group $\mathcal{G}$ in the general case, we should like to have a closer look, by way of example, at the two

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* If the original parameter p of the group is retained, e. g., as occurs in (67), then the equation (13) of the parameter group becomes $p'' = p + p' - p_0$.

special one-parameter linear homogeneous groups (2) and (1) of the Galilean and Lorentz transformations mentioned at the beginning.

The coefficients $b_{ik}(q)$ of group (2) of the Galilean transformations are :

(84)
$$\begin{cases} b_{11}(q) \equiv 1\ , & b_{12}(q) \equiv 0\ , \\ b_{21}(q) \equiv -\ q\ , & b_{22}(q) \equiv 1\ ; \end{cases}$$

from which we obtain for $q = 0$:

(44b)
$$\begin{cases} b_{11}(0) = 1\ , & b_{12}(0) = 0\ , \\ b_{21}(0) = 0\ , & b_{23}(0) = 1\ , \end{cases}$$

in agreement with the equations (44a) . It then follows from (84) that

(85)
$$\begin{cases} b_{11}'(q) \equiv 0\ , & b_{12}'(q) \equiv 0\ , \\ b_{21}'(q) \equiv -1, & b_{22}'(q) \equiv 0\ , \end{cases}$$

and, therefore, by (46a) :

(46b)
$$\begin{cases} \alpha_{11} = b_{11}'(0) = 0\ , & \alpha_{12} = b_{12}'(0) = 0\ , \\ \alpha_{21} = b_{21}'(0) = -1, & \alpha_{22} = b_{22}'(0) = 0\ , \end{cases}$$

so that equation (76) is satisfied. For the infinitesimal transformation (47a) and (52a), the result is thus:

(47b)
$$\delta t = 0\ , \quad \delta x = -\ t\ \delta\ q$$

and

(52b)
$$\delta w = -\ \delta q\ ;$$

and finally we obtain according to (57a) :

(57b)
$$\theta = 0\ ,$$

so that the two preferred velocities c_1, c_2 become equal to one another; that is, specifically

(86)
$$c_1 = c_2 = \infty\ ,$$

while the finite equation (73a) for the transformation of the velocity is transformed into

(73b)
$$w' = w - q$$

For the group (1) of the Lorentz transformations, the coefficients $b_{ik}(q)$ are given by :

(87)
$$\begin{cases} b_{11}(q) \equiv \dfrac{1}{\sqrt{1-\dfrac{q^2}{c^2}}}, & b_{12}(q) \equiv \dfrac{-\dfrac{q}{c^2}}{\sqrt{1-\dfrac{q^2}{c^2}}}, \\ b_{21}(q) \equiv \dfrac{-q}{\sqrt{1-\dfrac{q^2}{c^2}}}, & b_{22}(q) \equiv \dfrac{1}{\sqrt{1-\dfrac{q^2}{c^2}}} \end{cases}$$

from which, for $q = 0$, the equations (44a) again result. For the derivatives b_{ik}' of these coefficients, we find:

(88)
$$\begin{cases} b_{11}'(q) \equiv \dfrac{\dfrac{q}{c^2}}{\left(1-\dfrac{q^2}{c^2}\right)^{3/2}}, & b_{12}'(q) \equiv \dfrac{-\dfrac{1}{c^2}}{\left(1-\dfrac{q^2}{c^2}\right)^{3/2}}, \\ b_{21}'(q) \equiv \dfrac{-1}{\left(1-\dfrac{q^2}{c^2}\right)^{3/2}}, & b_{22}'(q) \equiv \dfrac{\dfrac{q}{c^2}}{\left(1-\dfrac{q^2}{c^2}\right)^{3/2}}, \end{cases}$$

and, from this results, using (46a):

(46c)
$$\begin{cases} \alpha_{11} = b_{11}'(0) = 0, & \alpha_{12} = b_{12}'(0) = -\dfrac{1}{c^2}, \\ \alpha_{21} = b_{21}'(0) = -1, & \alpha_{22} = b_{22}'(0) = 0, \end{cases}$$

where equation (76) is again satisfied. The equations (47a) and(52a) for the infinitesimal transformation now become

(47c)
$$\delta t = -\frac{x}{c^2}\,\delta q\,, \quad \delta x = -t\delta q\,,$$

and

(52c)
$$\delta w = -\left(1-\frac{w^2}{c^2}\right)\delta q$$

so that we obtain the following values for the two preferred velocities c_1, c_2;

(89)
$$c_1 = -c, \quad c_2 = +c$$

while, by (57a), θ becomes

(57c)
$$\theta = \frac{4}{c^2}$$

Finally, we obtain, from (73a) for the transformation of the velocity w, the finite equation :

(73a)
$$w' = \frac{w - q}{1 - \frac{q}{c^2} w} .$$

VI.

17. We now proceed to establish the general equations of the one-parameter linear homogeneous group $\mathcal{G}$; i. e., to determine the coefficients $b_{ik}(q)$ in (43a).

From a comparison of the two equations (51a) and (73a), which must agree with one another, it follows that the four coefficients:

(90)
$$\begin{cases} b_{11}(q), & b_{12}(q), \\ b_{21}(q), & b_{22}(q) . \end{cases}$$

must be proportional to the four quantities

(91)
$$\begin{cases} 1 + (\alpha_{11} - \alpha_{22})q, & \alpha_{12}q, \\ \quad - q, & 1 \end{cases}$$

where the factor of proportionality which is still to be determined can only be a function of q alone, which we shall designate by $\omega(q)$, so that then

(92)
$$\begin{cases} b_{11}(q) \equiv \omega(q) \, . \, [1 + (\alpha_{11} - \alpha_{22})q], & b_{12}(q) \equiv \omega(q) \, . \, \alpha_{12}q, \\ b_{21}(q) \equiv \omega(q) \, . \, (- q), & b_{22}(q) \equiv \omega(q) \end{cases}$$

whereby, moreover, identity (75) is also satisfied. By substitution of the quantities (92) into the equations (43a), we obtain these in the form

(93)
$$\begin{cases} t' = \omega(q)\{[1 + (\alpha_{11} - \alpha_{22})q]t + \alpha_{12}q\, x \} , \\ x' = \omega(q)\{ - q\, t + x\} , \end{cases}$$

where the function $\omega(q)$ is not determined as yet.

18. By means of equations (93), we can draw inferences concerning the kinematic significance of the factor $\omega(q)$, even before we have determined its form. Namely, if we examine a material point M , which moves on the x-axis with a constant velocity w with respect to the system S and which is found at the point $x = a$ at the time $t = 0$, then its motion with respect to S is given by the equation

(94)
$$x = a + w\, t$$

Now, in order to find the equation of motion of M with respect to a system S' moving towards S with velocity q, we solve the equations (93) for t and x, whereby we find the equations

$$(95)\quad \begin{cases} t = \dfrac{t' - \alpha_{12}\, q\, x'}{\omega(q)[1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2]}, \\[2ex] x = \dfrac{q\, t' + [1 + (\alpha_{11} - \alpha_{22})q]x'}{\omega(q)[1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2]} \end{cases}$$

for the inverse transformation of (92), and substitute the expressions (95) which have been found in (94). Thereby we next obtain

$$q\, t' + [1 + (\alpha_{11} - \alpha_{22})q]x'$$
$$= a[1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2]\, \omega(q) + \omega t' - \alpha_{12}q\, \omega\, x',$$

so, when we solve this equation for x', we get

$$(96)\quad \begin{cases} x' = a\, \dfrac{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}qw} \cdot \omega(q) \\[2ex] \quad + \dfrac{w - q}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}qw}\, t', \end{cases}$$

or

$$(97)\quad x' = a' + w'\, t',$$

where

$$(98)\quad a' = a\, \frac{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}\, q^2}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}qw} \cdot \omega(q)$$

signifies the value of x' at the time $t' = 0$ and w' is the velocity of M with respect to the system S' given by (73a).

We now consider two material points M_1 and M_2, whose space-time coordinates are t_1, x_1 and t_2, x_2 in the stationary system and which move with the same constant velocity w. If then at time

$$t_1 = t_2 = 0$$

the positions of M_1 and M_2 are given by

$$x_1 = a_1, \quad x_2 = a_2,$$

then the equations of motion of these two points with respect to the system S are:

(99) $$x_1 = a_1 + w\,t_1, \qquad x_2 = a_2 + w\,t_2,$$

while their equations of motion with respect to the system S' moving toward S with velocity q are :

(100) $$x_1' = a_1' + w'\,t_1', \qquad x_2' = a_2' + w'\,t_2'$$

where t_1', x_1' and t_2', x_2' signify the space-time coordinates of M_1 and M_2 measured in system S'; further, by (98):

(101) $$\begin{cases} a_1' = a_1 \dfrac{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}qw} \cdot \omega(q), \\[2ex] a_2' = a_2 \dfrac{1 + (\alpha_{11} - \alpha_{12})q + \alpha_{12}q^2}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}\,qw} \cdot \omega(q) \end{cases}$$

and the velocity w' of the point with respect to S' is again given by (73a) .

Since both points M_1 and M_2 move with the same velocity w on the x-axis, we may think of them as the end points of a rigid rod, the length l of which, measured in the system S, we obtain as the distance of two positions of M_1 and M_2, taken simultaneously with respect to S , if in (99) we let:

$$t_1 = t_2$$

and subtract the first equation from the second:

(102) $$l = x_2 - x_1 = a_2 - a_1 .$$

Similarly, when we use the relation

$$t_1' = t_2'$$

in equations (100), we find the following value for the length l' of the rod measured in system S' :

(103) $$l' = x_2' - x_1' = a_2' - a_1' ,$$

So, by (101) and (102),

(104) $$l' = \frac{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}\,q\,w} \cdot \omega(q) \cdot l .$$

Finally, if we assume that the rod is a rest with respect to the system S'; that, then, $w' = 0$, so it moves, by (69), with velocity $w = q$ with respect to the system S. Then, from (104)

(105) $$l' = \omega(q) \,.\, l$$

and, consequently:

The function $\omega(q)$ signifies that factor with which one must multiply the length l measured in the stationary system S of a rigid rod moving uniformly with velocity $w = q$ with respect to S, in order to obtain its length l' in that system S' with respect to which it is at rest.

The factor $\omega(q)$ is designated as "contraction".

19. Finally, to determine the form of the function $\omega(q)$ we combine the transformation (93) belonging to the parameter value q, which transforms the pair t, x into t', x', with a second transformation of the group $\mathcal{G}$:

(106) $$\begin{cases} t'' = \omega(q')\{[1 + (\alpha_{11} - \alpha_{22})q']t' + \alpha_{12}\, q'\, x'\}, \\ x'' = \omega(q')\{-\, q'\, t' + x'\} \end{cases}$$

which belongs to the parameter value q' and which transforms t', x' into t'', x''. From the group characteristic of the transformation (93) then follows that the resulting transformation, which transforms t, x directly into t'', x'' must be of the form

(107) $$\begin{cases} t'' = \omega(q'')\{[1 + (\alpha_{11} - \alpha_{22})q'']t + \alpha_{12}q''\, x\}, \\ x'' = \omega(q'')\{-\, q''\, t + x\} \end{cases}$$

where the parameter q'' is given by equation (80) as a function of q and q'.

If the combination of the two transformations (93) and (106) is actually carried out, one obtains taking into account equation (80):

(108) $$\begin{cases} t'' = (1 - \alpha_{12}\, qq')\omega(q)\omega(q')\{[1 + (\alpha_{11} - \alpha_{22})q'']t + \alpha_{12}q''\, x\}, \\ x'' = (1 - \alpha_{12}\, qq')\omega(q)\omega(q')\{-\, q''\, t + x''\}, \end{cases}$$

and from this follows, by comparison with (107):

(109) $$\omega(q'') = (1 - \alpha_{12}\, q\, q')\, \omega(q)\, \omega\,(q'),$$

that is, by (80):

(110) $$\omega\left(\frac{q + q' + (\alpha_{11} - \alpha_{22})q\, q'}{1 - \alpha_{12}\, q\, q'}\right) = (1 - \alpha_{12}\, q\, q')\omega(q)\omega(q') \,.$$

This is a functional equation, with the help of which the function $\omega(q)$ may be determined. For this purpose, let us differentiate (110) with respect to q' and then set $q' = 0$, whereby, we obtain:

$$\omega'(q)[1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2] = \omega(q)[\omega'(0) - \alpha_{12}\omega(0)q] \; . \tag{111}$$

Now, according to the last equation in (92)

$$\omega(q) \equiv b_{22}(q),$$

so that, using (44a) and (46a) we obtain for the contraction $\omega(q)$ the conditions

$$\omega(0) = b_{22}(0) = 1 \tag{112}$$

and,

$$\omega'(0) = b_{22}'(0) = \alpha_{22} \tag{113}$$

by means of which the differential equation

$$\omega'(q)[1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2] = \omega(q)[\alpha_{22} - \alpha_{12}\, q] \tag{114}$$

results, using (111) and initial conditions(112). It follows from (114) that

$$\frac{\omega'(q)}{\omega(q)} = \frac{\alpha_{22} - \alpha_{12}q}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2} \tag{115}$$

and, therefore

$$\int_0^q \frac{\omega'(q)}{\omega(q)}\, dq = \int_0^q \frac{\alpha_{22} - \alpha_{12}\, q}{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2}\, dq \; . \tag{116}$$

If the integrals on both sides are evaluated and the resulting equation for $\omega(q)$ is solved one finally finds the expression

$$\omega(q) = \frac{1}{\sqrt{1 + (\alpha_{11} - \alpha_{22})q + \alpha_{12}q^2}} \left[\frac{1 + \dfrac{\alpha_{11} - \alpha_{22} + \sqrt{\theta}}{2}\, q}{1 + \dfrac{\alpha_{11} - \alpha_{22} - \sqrt{\theta}}{2}\, q} \right]^{\frac{\alpha_{11}+\alpha_{22}}{2\sqrt{\theta}}} , \tag{117}$$

for the contraction. This indeed satisfies condition (113) also.

With this, the finite equations (93) of the general one-parameter linear homogeneous group which is generated by means of the infinitesimal transformation (47) under the supposition (55), are fully determined.*

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*Equation (117) is not affected by a change in sign of $\sqrt{\theta}$ either (Compare the footnote to No. 12).

20. For the Galilean group, one finds, specifically, using (46b):

(117a) $$\omega(q) = 1$$

and for the Lorentz group, using (46c):

(117b) $$\omega(q) = \frac{1}{\sqrt{1 + \alpha_{12}q^2}} = \frac{1}{\sqrt{1 - \frac{q^2}{c^2}}}$$

This conforms with equations (2) and (1) .

VII.

21. We now proceed to utilize postulate B of our introduction and to examine which of the ∞^3 transformation groups which are given by equations (93) and (117) lead to a contraction $\omega(q)$, which is an even function of the velocity q; i.e., which does not favor either of the two directions of the x-axis.

For this, it is certainly necessary and sufficient that the differential quotients of odd order of the function $\omega(q)$ vanish at the point $q = 0$. Specifically, we have:

(118) $$\left(\frac{d\omega}{dq}\right)_{q=0} = \omega'(0) = 0, \qquad \left(\frac{d^3\omega}{dq^3}\right)_{q=0} = \omega'''(0) = 0$$

We thus, determine the quantities $\omega(0)$, $\omega'(0)$, $\omega''(0)$, $\omega'''(0)$.

The first two are given by equations (112) and (113); we obtain the remaining ones in the simplest fashion by repeated differentiation of equation (114). If we simultaneously set $q = 0$, this yields

(119) $$\omega''(0) + (\alpha_{11} - 2\alpha_{22})\omega'(0) + \alpha_{12}\omega(0) = 0,$$

(120) $$\omega'''(0) + (2\alpha_{11} - 3\alpha_{22})\omega''(0) + 4\alpha_{12}\omega'(0) = 0.$$

It follows from this that

(121) $$\omega''(0) = - \alpha_{12} - \alpha_{22}(\alpha_{11} - 2\alpha_{22}),$$

(122) $$\omega'''(0) = 2\alpha_{11}\alpha_{12} + \alpha_{22}(2\alpha_{11}^2 - 7\alpha_{12} - 7\alpha_{11}\alpha_{22} + 6\alpha_{22}^2) .$$

From the first of the equations (118) it follows, in conjunction with equation (113), that

(123) $$\alpha_{22} = 0$$

and in conjunction with (122):

(124) $$\alpha_{11}\alpha_{12} = 0 .$$

Hence, the equations (123) and (124) must necessarily be satisfied when the transformation equations are to yield a contraction which satisfies postulate B . We shall soon see that the existence of equations (123) and (124) is also sufficient for this.

22. Namely, there are, according to equation (124), three subcases:

(125a) $$1.\ \alpha_{11} = 0, \quad \alpha_{12} = 0 ,$$

(125b) $$2.\ \alpha_{11} = 0, \quad \alpha_{12} \neq 0 ,$$

(125c) $$3.\ \alpha_{11} \neq 0, \quad \alpha_{12} = 0 .$$

To each of these subcases corresponds a certain type of transformation equations which satisfy our postulates A and B .

As may be seen from equation (93) in conjunction with (117) and (117a), the first subcase yields the group of Galilean transformations; the second subcase, the group of Lorentz transformations, as may be seen from equations (93) in conjunction with equation (117) and (117b).

23. The third subcase however, leads to a group which has not yet been treated. It follows from equation (117) in conjunction with (123) and (125c):

(126) $$\omega(q) = 1$$

and from equation (93):

(127) $$\begin{cases} t' = (1 + \alpha_{11} q)t, \\ x' = - q\ t + x\ . \end{cases}$$

The preferred velocities have the values:

(128) $$c_1 = - \frac{1}{\alpha_{11}} , \quad c_2 = \infty ,$$

since these are the roots of the quadratic equation (54) when its coefficients satisfy the conditions (76), (123) and (125c). The transformation equations (127) may then be written in the form:

(129) $$\begin{cases} t' = (1 - \frac{q}{c_1})\ t, \\ x' = - q\ t + x \end{cases}$$

The regulation of the clock represented by this transformation may now be interpreted physically in a manner similar to that which Einstein [5] used for the Lorentz transformation.

Suppose that at time $t = 0$, a ray of light emanates from the origin in the positive direction and propagates with velocity c_1. Now, when a body moves with the velocity q, then the light has the velocity $c_1 - q$ (in the time of the system at rest) with respect to this body. Now, if we wish that the velocity of the light ray with respect to the moving body should still be c_1 we can attain this by changing the rate of the clocks in the ratio c_1 to (c_1-q). But thereby we introduce in the moving body a time t', which is given by:

$$t' = \frac{c_1 - q}{c_1} t,$$

i.e., by the first of equations (129). This regulation of time corresponds to the Doppler principle. We will therefore designate the equations (129) as Doppler transformations.

The Doppler transformation is essentially different from the Lorentz transformation in that for a body moving with velocity q the same time prevails at all positions; there exists no local time, and, what is more important: when we have made provisions for the regulation for the light rays propagating in the direction of the positive x-axis, which therefore now possess the same velocity c_1 in all moving bodies (here we presuppose $c_1 > 0$), then the velocity of propagation of light rays which propagate in the negative direction with the velocity c_1 with respect to the system at rest, for this reason is not yet the same with respect to all moving bodies.

For just because c_1 is a preferred velocity, $(-c_1)$ is not such a velocity (a priori). According to equation (128), this would only be the case for $c_1 = \infty$, i.e., $\alpha_{11} = 0$. But, in that case, we are dealing with a Galilean transformation.

On the other hand, for the Lorentz transformation, as a result of equations (76), (123) and (125b), we have

$$c_1 = -c_2 = c \quad \text{(comp. also equation (89))}.$$

We may therefore summarize the result of our investigation in the following manner:

Among all transformation equations which correspond to one-parameter linear homogeneous groups, there exist three types for which the amount of contraction does not depend on the direction of motion in absolute space: Among these, only one type has as its consequence an actual contraction of lengths, namely, the Lorentz transformation [equation (1)], while the other two types, the Galilean and the Doppler transformations [equations (2), (129) respectively], leave their lengths unchanged. For the Lorentz transformation, the velocity of light in all moving systems has, for any arbitrary direction of propagation, the same finite value c. For the Doppler transformation, however, this is true only for propagation in one direction; for the Galilean transformation, only if the velocity of light were infinite.

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References

1. A. EINSTEIN: Jahrb. d. Rad. u. Elektr. 4, p.411, ff. (1907). A more detailed version of Einstein's derivation is found in the paper: PH. FRANK: Sitzungsber, d.k. Akad. d. Wiss. in Wien., Math.-Phys. Kl, 118, Abt. IIa., p. 421, ff, (1909).
2. PH. FRANK; H. ROTHE: Sitzungsb. d.k.Akad. d.Wiss. in Wien. Math.-Nat., Kl., 119, Abt. IIa, p.615 ff. (1910).
3. W. v. IGNATOWSKY: Berichte d. Deutsch. Physik. Ges., p. 788 ff. (1910) and Arch. f. Math. u. Phys., 17, p. 1 ff. (1910).
4. S. LIE AND G. SCHEFFERS: p. 32, Theorem 2.
5. A. EINSTEIN: Ann. d. Phys., 17, p. 891 ff. (1905).